The Internal Model Principle for Linear Multivariable Regulators

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ABSTRACT

Necessary structural criteria are obtained for linear multivariable regulators which retain loop stability and output regulation in the presence of small perturbations, of specified types, in system parameters. It is shown that structural stability thus defined requires feedback of the regulated variable, together with a suitably reduplicated model, internal to the feedback loop, of the dynamic structure of the exogenous reference and disturbance signals which the regulator is required to process. Necessity of these structural features constitutes the 'internal model principle'.

1. Introduction. The problem of synthesizing linear multivariable regulators possessing structural stability with respect to small perturbations in system parameters was considered in [1, Chapter 8]. The synthesis exploited feedback together with a suitably reduplicated model, internal to the feedback loop, of the dynamic structure of the exogenous reference and disturbance signals which the system was required to process. In this paper it is shown that such structure of the regulator is actually necessary. This general result, which no doubt is of broader validity than is proved here, we call the *internal model principle*.

We remark that plausibility arguments in support of the 'internal model' idea have been presented by Kelley [2]. In addition Davison [3] has discussed a problem similar to that addressed in this paper. The present treatment is quite different in method of approach.

The system under consideration is modeled as follows. The plant is described by the linear time-invariant vector differential equation

$$\dot{x}_1 = A_1 x_1 + A_3 x_2 + B_1 u, \tag{1}$$

where x_1 is the state vector of the plant and u is the vector of control inputs. The vector x_2 satisfies the equation

$$\dot{x}_2 = A_2 x_2 \tag{2}$$

and represents reference and/or disturbance signals which the regulator is to be

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designed to process. The vector z to be regulated (e.g. tracking errors and/or deviations from set-point values) is given by

$$z = D_1 x_1 + D_2 x_2. \tag{3}$$

The vector y of outputs measurable by the controller is given by

$$y = C_1 x_1 + C_2 x_2. \tag{4}$$

Typically the vector $A_3 x_2$ in (1) represents plant disturbances, and the vector $D_2 x_2$ in (3) represents reference signals which the controlled output vector $-D_1 x_1$ is required to track.

The control action is generated by a linear time-invariant compensator with input $y(\cdot)$ and output $u(\cdot)$, according to

$$\dot{x}_c = A_c x_c + B_c y \tag{5a}$$

$$u = F_c x_c + F y. \tag{5b}$$

Here x_c is the state vector of the compensator.

The vectors x_1, x_2, u, z, y, x_c belong to fixed real linear spaces

$$\mathcal{X}_1, \mathcal{X}_2, \mathcal{U}, \mathcal{Z}, \mathcal{Y}, \mathcal{X}_c \tag{6}$$

of finite dimension n_1, n_2, m, q, p, n_c respectively. The time-invariant linear maps in (1) through (5) are defined on the appropriate spaces as follows:

$$\begin{array}{llll} A_i : \mathcal{X}_i \to \mathcal{X}_i, & D_i : \mathcal{X}_i \to \mathcal{Z}, & C_i : \mathcal{X}_i \to \mathcal{Y}, & i = 1,2 \\ A_3 : \mathcal{X}_2 \to \mathcal{X}_1, & B_1 : \mathcal{U} \to \mathcal{X}_1, & A_c : \mathcal{X}_c \to \mathcal{X}_c, & \\ B_c : \mathcal{Y} \to \mathcal{X}_c, & F_c : \mathcal{X}_c \to \mathcal{U}, & F : \mathcal{Y} \to \mathcal{U}. & \end{array}$$

The signal flow graph of the system is shown in Fig. 1. ($sI - A_1$ is written $s - A_1$, etc.).

From Fig. 1 it is seen that the plant and compensator together form a loop. The state space of the loop, $\mathcal{X}_L$, is defined to be the external direct sum of $\mathcal{X}_1$ and $\mathcal{X}_c$:

$$\mathcal{X}_L = \mathcal{X}_1 \oplus \mathcal{X}_c.$$

The state vector of the loop will be written

$$x_L = \begin{bmatrix} x_1 \\ x_c \end{bmatrix} \in \mathcal{X}_L. \tag{7}$$

Then, by combining (1), (4) and (5), we see that the loop is described by

$$\dot{x}_L = A_L x_L + B_L x_2, \tag{8}$$

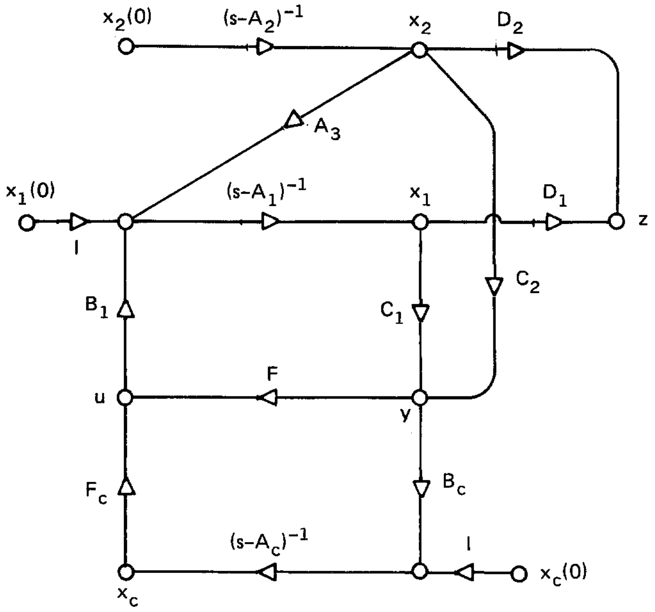


Figure 1. Overall system signal flow graph.

where

$$A_L=\begin{bmatrix} A_1+B_1FC_1 & B_1F_c \\ B_cC_1 & A_c \end{bmatrix}, \quad B_L=\begin{bmatrix} A_3+B_1FC_2 \\ B_cC_2 \end{bmatrix}. \tag{9}$$

In (8) A_L is a map $\mathfrak{X}_L \to \mathfrak{X}_L$ which can be written as the matrix shown in (9) because of the convention (7). In addition define $D_L : \mathfrak{X}_L \to \mathfrak{Z}$ by

$$D_L=[D_1 \quad 0], \tag{10}$$

so that

$$D_L x_L = D_1 x_1.$$

The output to be regulated is thus

$$z = D_L x_L + D_2 x_2. \tag{11}$$

The composite system is now described by (2), (8) and (11); the signal flow graph is shown in Fig. 2.

The purpose of the controller is twofold: to stabilize the loop and regulate the output. By *loop stability* is meant that $x_L(t) \to 0$ as $t \to \infty$ for all $x_L(0)$ with $x_2(0)=0$. From (8) this is evidently equivalent to stability of A_L, that is

$$\sigma(A_L) \subset \mathbf{C}^- = \{z \in \mathbf{C} : \operatorname{Re} z < 0\}.$$

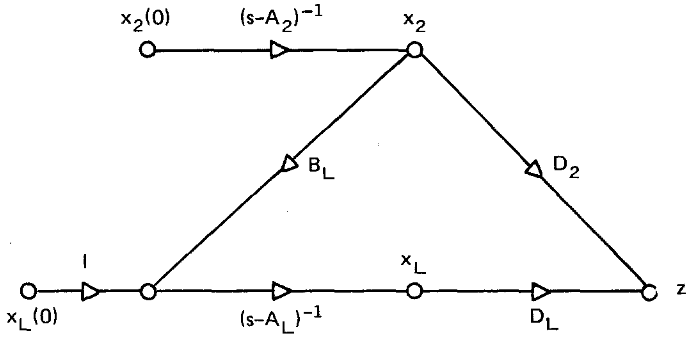


Figure 2. Signal flow graph showing loop.

By *output regulation* is meant that $z(t) \to 0$ as $t \to \infty$ for all $x_L(0)$ and $x_2(0)$.
We may assume at the outset that A_2 is totally unstable, that is

$$\sigma(A_2) \subset \mathbf{C}^+ = \{z \in \mathbf{C} : \operatorname{Re} z \geqslant 0\}, \tag{12}$$

because any stable exogenous modes can be included in the plant description since they affect neither loop stability nor output regulation*. Also we may assume that

$$[C_1, C_2] \text{ is epic;} \tag{13}$$

otherwise replace $\mathscr{Y}$ by Im C_1 + Im C_2. Finally we may assume that

$$D_1 \text{ is epic} \tag{14}$$

since clearly a necessary condition for output regulation is Im $D_2 \subset$ Im D_1 and hence (14) is achieved by setting $\mathscr{Z}$ = Im D_1 + Im D_2 = Im D_1. Throughout the remainder of the paper (12), (13) and (14) are standing assumptions which will not be reiterated in the statements of our results.

The vector spaces in (6) are assumed to have fixed bases. The maps in (1) to (5) then have matrix representations referred to these bases. These matrices will be denoted by the same letter as the corresponding maps. The matrix A_1 can then be regarded as a data point in $\mathbf{R}^N$ where $N = n_1^2$. Similarly (A_1, B_1) can be regarded as a data point in $\mathbf{R}^N$ with $N = n_1^2 + n_1 m$. Let $\mathfrak{p}$ be such a data point in $\mathbf{R}^N$, and assign to $\mathbf{R}^N$ its usual topology. To say that the *synthesis is structurally stable at* $\mathfrak{p}$ means that loop stability and output regulation hold everywhere throughout some open neighborhood (nbhd) of $\mathfrak{p}$ in $\mathbf{R}^N$. If A_L is stable and its elements are perturbed slightly the resulting matrix is still stable. Thus structural

*The specific partition $\mathbf{C} = \mathbf{C}^- \cup \mathbf{C}^+$ of the complex plane is not crucial to the development, and could be replaced by an arbitrary 'symmetric partition' in the sense of [1], with $\mathbf{C}^-$ an open subset of the complex plane.

stability at $\mathfrak{p}$ is equivalent to the conditions:

(i) A_L is stable
(ii) output regulation holds throughout a nbhd of $\mathfrak{p}$ in $\mathbf{R}^N$.

The main objective of this paper is to determine the necessary controller structure so that the resulting synthesis is structurally stable at a given data point. The data point in question will depend on the theorem discussed, as the intent is partly to establish to what system data the controller may be critically sensitive. The data point will not include elements from A_2, since it is assumed that a fixed class of exogenous signals is to be processed, this class being specified a priori in a statement of the design objective. Nor will the data point include C_1, C_2, D_1 or D_2; the elements in these matrices are assumed fixed either by definitional relations or by the precision of the physical sensors and error comparators being modeled.

The principal concepts of readability, the internal model, and feedback are introduced in Section 2. That a structurally stable synthesis must incorporate these features is proved in the sections which follow.

Notation. The real and complex fields are denoted respectively by $\mathbf{R}$ and $\mathbf{C}$. The real part of a complex number is written Re.

If $\mathcal{X}$ is a linear space, $d(\mathcal{X})$ is its dimension. While $\mathcal{X}$ etc. is defined initially over $\mathbf{R}$, the complexification of $\mathcal{X}$ etc. will be denoted by the same symbol and introduced freely. If $A : \mathcal{X} \to \mathcal{X}$ is a linear map, $\sigma(A)$ is its complex spectrum. If $B : \mathcal{U} \to \mathcal{X}$, Im B or $\mathcal{B}$ denotes the image of B and Ker B its kernel. The controllable subspace of (A,B) is $\langle A|\mathcal{B}\rangle = \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B}$ where $n = d(\mathcal{X})$. If $\mathcal{V} \subset \mathcal{U}$ then $B|\mathcal{V}$ is the restriction of B to $\mathcal{V}$ and is a map $\mathcal{V} \to \mathcal{X}$; whereas if $A\mathcal{V} \subset \mathcal{V} \subset \mathcal{X}$ then $A|\mathcal{V} : \mathcal{V} \to \mathcal{V}$. Linear space isomorphism is denoted by $\approx$.

If $n \geqslant 1$ is an integer $\underline{n}$ is the set $\{1,\ldots,n\}$. Definitional equality is written $:=$.

2. Principal Concepts. The compensator is restricted to process only the measurable output y. It will be shown that a synthesis can be structurally stable only if the compensator has access to the regulated variable z; that is, only if the value of $y(t)$ always determines that of $z(t)$. This motivates the definition: z is *readable from* y if there exists a map Q: $\mathcal{Y} \to \mathcal{Z}$ such that $z = Qy$. This is equivalent to the condition that $[D_1, D_2]$ factors through $[C_1, C_2]$, namely

$$\operatorname{Ker}[C_1, C_2] \subset \operatorname{Ker}[D_1, D_2]. \tag{15}$$

If (15) holds we shall say that the pair $([C_1, C_2], [D_1, D_2])$ is *readable*.

An alternative way of describing readability is as follows. Suppose (15) holds. Then $d(\mathcal{Y}) \geqslant d(\mathcal{Z})$ by virtue of (13) and (14), and $\mathcal{Y}$ can be defined according to

$$\mathcal{Y} = \mathcal{W} \oplus \mathcal{Z}, \tag{16}$$

for a suitable linear space $\mathcal{W}$. Then

$$[C_1, C_2] = \begin{bmatrix} E_1 & E_2 \\ D_1 & D_2 \end{bmatrix} \tag{17}$$

for some maps $E_i : \mathcal{X}_i \to \mathcal{W}$, $i \in \underline{2}$. Defining $w = E_1 x_1 + E_2 x_2$ we have

$$y = \begin{bmatrix} w \\ z \end{bmatrix} \in \mathcal{W} \oplus \mathcal{Z}.$$

The map Q is now the natural projection $\mathcal{W} \oplus \mathcal{Z} \to \mathcal{Z}$.

Corresponding to (16) the maps F and B_c can be written

$$F = [F_w, F_z], \quad B_c = [B_{cw}, B_{cz}], \tag{18}$$

that is $F_w = F|\mathcal{W}$, $F_z = F|\mathcal{Z}$, etc. When the representations (17), (18) are inserted in Fig. 1 the result is as shown in Fig. 3.

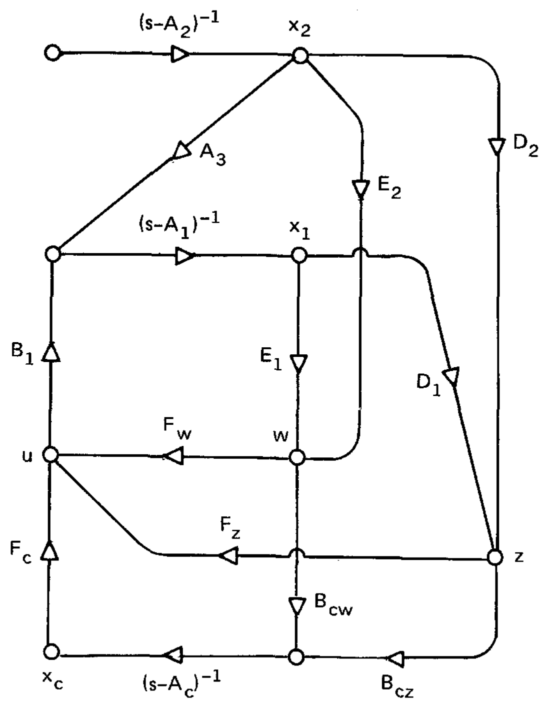


Figure 3. Synthesis in which z is readable from y.

To define the internal model we first recall that the invariant factors (i.f.) of a map $A:\mathfrak{X}\to\mathfrak{X}$ are the minimal polynomials of the cyclic components in a rational canonical decomposition of $\mathfrak{X}$ relative to A. We say that *A incorporates an internal model of A_2* if the minimal polynomial (m.p.) of A_2 divides at least $q=d(\mathfrak{Z})$ i.f. of A. The internal model is thus a q-fold reduplication in A of the maximal cyclic component of A_2.

The internal model can also be described in terms of Jordan decomposition. Recall that the Jordan decomposition of $\mathfrak{X}$ relative to a map $A:\mathfrak{X}\to\mathfrak{X}$ can be derived from its rational canonical decomposition by factoring the i.f. of A into powers of monomials of the form $s-\lambda$, where $\lambda\in\sigma(A)$. Precisely, for each distinct $\lambda\in\sigma(A)$ there exist an integer $t(\lambda)$ (which is unique) and A-invariant subspaces $\mathfrak{X}_\lambda^i$, $i\in \underline{t(\lambda)}$, such that

(i) $\mathfrak{X}=\bigoplus_{\lambda\in\sigma(A)}\ \bigoplus_{i\in\underline{t(\lambda)}}\mathfrak{X}_\lambda^i$

(ii) $A|\mathfrak{X}_\lambda^i$ is cyclic with m.p. $(s-\lambda)^{k(\lambda,i)}$, where $k(\lambda,i):=d(\mathfrak{X}_\lambda^i)$. There is a basis for $\mathfrak{X}_\lambda^i$ such that $A|\mathfrak{X}_\lambda^i$ is represented by the Jordan matrix

$$J_{k(\lambda,i)}(\lambda):=\begin{bmatrix}\lambda & 1 & & & \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1\\ & & & & \lambda\end{bmatrix}:k(\lambda,i)\times k(\lambda,i).$$

(iii) The m.p. of A is

$$\prod_{\lambda\in\sigma(A)}(s-\lambda)^{k(\lambda)}$$

where

$$k(\lambda):=\max\left\{k(\lambda,i):i\in\underline{t(\lambda)}\right\}.$$

The polynomials $(s-\lambda)^{k(\lambda,i)}$ are the elementary divisors (e.d.) of A. The $\mathfrak{X}_\lambda^i$ are the *prime* subspaces corresponding to λ. The integer $t(\lambda)$ can be computed as

$$t(\lambda)=d[\operatorname{Ker}(A-\lambda)].$$

The integer $k(\lambda)$ is the degree of the factor $s-\lambda$ in the m.p. of A.

From now on $k(\lambda)$ will denote specifically the degree of the factor $s-\lambda$ in the m.p. of A_2, $\lambda\in\sigma(A_2)$. Consider a Jordan decomposition of $\mathfrak{X}$ relative to A. Then A incorporates an internal model of A_2 if and only if for each $\lambda\in\sigma(A_2)$ there are independent prime subspaces $\mathfrak{X}_\lambda^i$ ($i\in\underline{q}$) of $\mathfrak{X}$ such that $d(\mathfrak{X}_\lambda^i)\geqslant k(\lambda)$. It is easy to see that this characterization agrees with the previous definition. Indeed, the m.p. of A_2 divides q i.f. of A if and only if for each $\lambda\in\sigma(A_2)$,

$(s-\lambda)^{k(\lambda)}$ divides q i.f. of A; this is equivalent to the condition that $(s-\lambda)^{\ell_1},\ldots,(s-\lambda)^{\ell_q}$ are e.d. of A for q integers $\ell_i \geqslant k(\lambda)$; and this in turn is equivalent to the existence of the prime subspaces $\mathcal{X}_\lambda^i$.

Next we recall the following characterizations of controllability and observability. Let $A:\mathcal{X}\to\mathcal{X}$, $B:\mathcal{U}\to\mathcal{X}$, $C:\mathcal{X}\to\mathcal{Y}$. Then (A,B) is controllable if and only if

$$\mathcal{X} = \operatorname{Im}(A-\lambda)+\operatorname{Im}B,\quad \lambda\in\sigma(A)$$

and (C,A) is observable if and only if

$$\operatorname{Ker}C\cap\operatorname{Ker}(A-\lambda)=0,\quad \lambda\in\sigma(A).$$

(See [1, Lemma 8.1] for a proof of the second assertion; the first follows by duality.)

Now let $\mathcal{R}_c\subset\mathcal{X}_c$ be an A_c-invariant subspace and let $P_c:\mathcal{X}_c\to\overline{\mathcal{X}}_c:=\mathcal{X}_c/\mathcal{R}_c$ be the canonical projection. Let $\overline{A}_c:\overline{\mathcal{X}}_c\to\overline{\mathcal{X}}_c$ be the map induced by A_c in $\overline{\mathcal{X}}_c$, i.e. $\overline{A}_c P_c = P_c A_c$. We say that *the compensator incorporates an internal model of* A_2 provided

(IM) $\mathcal{R}_c$ exists (as above) such that the map $\overline{A}_c$ incorporates an internal model of A_2, in the sense of the definition given earlier.

Next assume that, in addition to (IM),

$$\text{(i)}\quad \operatorname{Ker}F_c\cap\operatorname{Ker}(A_c-\lambda)=0,\quad \lambda\in\sigma(A_2).$$

Then we say that *the internal model is observable by u*. In view of the above characterization of observability, (i) is equivalent to the condition that the A_2-modes of A_c are observable by F_c.

Finally suppose that z is readable from y and that, in addition to (IM),

$$\text{(ii)}\quad \operatorname{Im}B_{cw}\subset\mathcal{R}_c \quad\text{and}$$

$$\overline{\mathcal{X}}_c=\operatorname{Im}\left(\overline{A}_c-\lambda\right)+\operatorname{Im}\overline{B}_{cz},\quad \lambda\in\sigma(A_2)$$

where $\overline{B}_{cz}:=P_cB_{cz}$. Then we say that *the internal model is controllable by z*. In view of the above characterization of controllability, (ii) is equivalent to the condition that the A_2-modes of $\overline{A}_c$ are controllable by $\overline{B}_{cz}$.

To see what these concepts mean in terms of signal flow assume there exists such a subspace $\mathcal{R}_c$ and write

$$\mathcal{X}_c=\mathcal{X}_{c1}\oplus\mathcal{X}_{c2}$$

where $\mathcal{X}_{c1}=\mathcal{R}_c$ and $\mathcal{X}_{c2}$ is an arbitrary complement. Corresponding to this decomposition write

$$A_c=\begin{bmatrix} A_{c1} & A_{c3}\\ 0 & A_{c2}\end{bmatrix},\quad B_{cw}=\begin{bmatrix} B_{cw1}\\ B_{cw2}\end{bmatrix},\quad B_{cz}=\begin{bmatrix} B_{cz1}\\ B_{cz2}\end{bmatrix},\quad F_c=[F_{c1},F_{c2}].$$

Then (IM) is equivalent to the condition that A_{c2} incorporates an internal model of A_2, and (ii) to the conditions $B_{cw2}=0$ and

$$\mathcal{X}_{c2}=\operatorname{Im}(A_{c2}-\lambda)+\operatorname{Im}B_{cz2},\quad \lambda\in\sigma(A_2).$$

Inserting this decomposition in Fig. 3 yields Fig. 4.

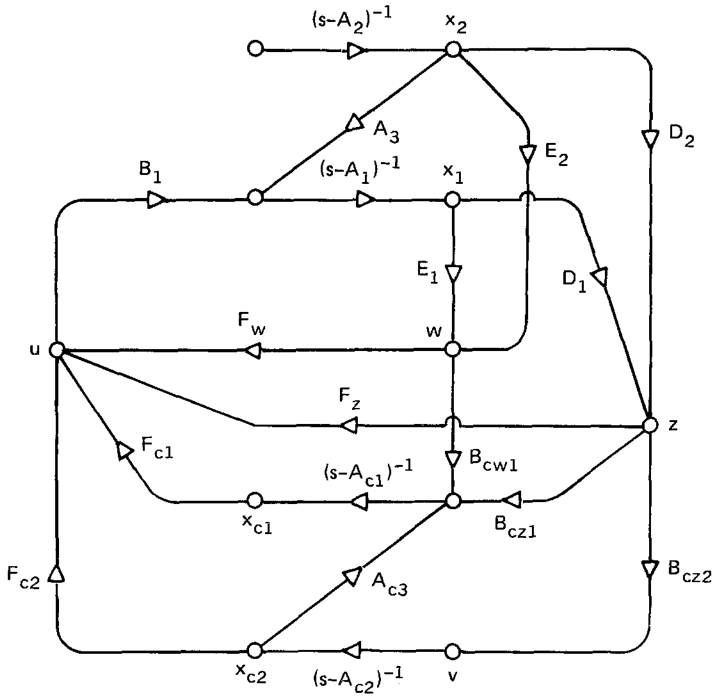


Figure 4. The canonical synthesis.

In Fig. 4 the signal flow from z to u can be thought of as an information transmission system: information is transmitted from the z terminal and received at the u terminal; the internal model is the channel. The controllability condition (ii) means that the internal model processes all the information transmitted from z. Similarly the observability condition (i) means that u receives all this processed information. In this sense the synthesis illustrated in Fig. 4 is a feedback synthesis.

3. Technicalities. In this section we derive a useful characterization of output regulation.

Lemma 1. *Assume loop stability. Then output regulation is equivalent to the existence of a map* $R:\mathcal{X}_2\to\mathcal{X}_L$ *such that*

$$A_LR-RA_2=B_L \tag{19}$$

and

$$D_L R = D_2. \tag{20}$$

Proof. From (12) and the stability of A_L,

$$\sigma(A_L) \cap \sigma(A_2) = \emptyset .$$

Hence the map $R \mapsto A_L R - RA_2$ is invertible and thus (19) defines R uniquely ([4, p. 225]).

Let $\mathcal{X}_s := \mathcal{X}_L \oplus \mathcal{X}_2$. From (8) and (11) the system is described by

$$\dot{x}_s = A_s x_s$$
$$z = D_s x_s$$

where

$$x_s := \begin{bmatrix} x_L \\ x_2 \end{bmatrix} \in \mathcal{X}_s$$

$$A_s := \begin{bmatrix} A_L & B_L \\ 0 & A_2 \end{bmatrix}, \quad D_s := [D_L, D_2].$$

Let $\mathcal{X}_s^+(A_s)$ denote the unstable subspace of A_s in $\mathcal{X}_s$ (see [1]). Then output regulation is equivalent to

$$\mathcal{X}_s^+(A_s) \subset \operatorname{Ker} D_s. \tag{21}$$

Let $Q: \mathcal{X}_s \to \mathcal{X}_s$ be defined as

$$Q := \begin{bmatrix} I_L & -R \\ 0 & I_2 \end{bmatrix}$$

where I_L (resp. I_2) is the identity on $\mathcal{X}_L$ (resp. $\mathcal{X}_2$). Then

$$Q^{-1} A_s Q = \begin{bmatrix} A_L & -A_L R + RA_2 + B_L \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_L & 0 \\ 0 & A_2 \end{bmatrix}$$

from (19). Thus

$$\mathcal{X}_s^+(A_s) = Q \operatorname{Im} \begin{bmatrix} 0 \\ I_2 \end{bmatrix} = \operatorname{Im} \begin{bmatrix} -R \\ I_2 \end{bmatrix}.$$

Therefore (21) is equivalent to (20). ■

The main results of this paper will be obtained by the following application of Lemma 1.

Proposition 1. *Let* $\lambda\in\sigma(A_2)$ *and let* $\mathcal{X}_{2\lambda}$ *be any prime subspace of* $\mathcal{X}_2$ *corresponding to* λ. *Set* $k=d(\mathcal{X}_{2\lambda})$ *and define the subspace* $\mathcal{S}\subset\mathcal{X}_1\oplus\mathcal{X}_c$ *as follows*:

$$\mathcal{S}=\{(x_1,x_c): B_cC_1x_1+(A_c-\lambda)x_c=0,\quad x_1\in\operatorname{Ker}D_1,$$

$$x_c\in\langle A_c|B_cC_1\operatorname{Ker}D_1\rangle+\operatorname{Im}(A_c-\lambda)^{k-1}\}. \tag{22}$$

Then structural stability at A_3 *implies* $d(\mathcal{S})\geqslant n_1$.

In (22), $\langle A_c|B_cC_1\operatorname{Ker}D_1\rangle$ is the controllable subspace of the pair $(A_c,B_cC_1|\operatorname{Ker}D_1)$.

Proof. First we write out equations (19) and (20) in detail. For $R:\mathcal{X}_2\to\mathcal{X}_L$ let

$$R=\begin{bmatrix} R_1\\ R_c\end{bmatrix}:\mathcal{X}_2\to\mathcal{X}_1\oplus\mathcal{X}_c.$$

Substitution of (9), (10) into (19), (20) yields

$$(A_1+B_1FC_1)R_1-R_1A_2+B_1F_cR_c=A_3+B_1FC_2 \tag{23a}$$

$$B_cC_1R_1+A_cR_c-R_cA_2=B_cC_2 \tag{23b}$$

$$D_1R_1=D_2. \tag{23c}$$

Let $A_{2\lambda}$, $A_{3\lambda}$, $C_{2\lambda}$, $D_{2\lambda}$, $R_{1\lambda}$ and $R_{c\lambda}$ denote respectively the restriction of A_2, A_3, C_2, D_2, R_1 and R_c to $\mathcal{X}_{2\lambda}$. Then (23) implies

$$(A_1+B_1FC_1)R_{1\lambda}-R_{1\lambda}A_{2\lambda}+B_1F_cR_{c\lambda}=A_{3\lambda}+B_1FC_{2\lambda} \tag{24a}$$

$$B_cC_1R_{1\lambda}+A_cR_{c\lambda}-R_{c\lambda}A_{2\lambda}=B_cC_{2\lambda} \tag{24b}$$

$$D_1R_{1\lambda}=D_{2\lambda}. \tag{24c}$$

As noted in Section 2 there is a basis for $\mathcal{X}_{2\lambda}$ such that $A_{2\lambda}$ has the matrix representation $J_k(\lambda)$. In this basis write

$$R_{1\lambda}=[r_{11},\dots,r_{1k}],\quad R_{c\lambda}=[r_{c1},\dots,r_{ck}]$$

$$A_{3\lambda}=[a_{31},\dots,a_{3k}],\quad C_{2\lambda}=[c_{21},\dots,c_{2k}]$$

$$D_{2\lambda}=[d_{21},\dots,d_{2k}].$$

Here the r_{1j} are the columns of $R_{1\lambda}$, etc. Clearly

$$R_{1\lambda}A_{2\lambda}=R_{1\lambda}J_k(\lambda)=[\lambda r_{11}, r_{11}+\lambda r_{12},\ldots,r_{1,k-1}+\lambda r_{1k}].$$

Thus (24a) is equivalent to the k vector equations

$$\begin{aligned}(A_1+B_1FC_1-\lambda)r_{11}+B_1F_cr_{c1}&=a_{31}+B_1Fc_{21}\\ -r_{11}+(A_1+B_1FC_1-\lambda)r_{12}+B_1F_cr_{c2}&=a_{32}+B_1Fc_{22}\\ &\vdots\\ -r_{1,k-1}+(A_1+B_1FC_1-\lambda)r_{1k}+B_1F_cr_{ck}&=a_{3k}+B_1Fc_{2k}.\end{aligned} \tag{25}$$

Similarly (24b) is equivalent to

$$\begin{aligned}B_cC_1r_{11}+(A_c-\lambda)r_{c1}&=B_cc_{21}\\ B_cC_1r_{12}-r_{c1}+(A_c-\lambda)r_{c2}&=B_cc_{22}\\ &\vdots\\ B_cC_1r_{1k}-r_{c,k-1}+(A_c-\lambda)r_{ck}&=B_cc_{2k}.\end{aligned} \tag{26}$$

Finally (24c) is equivalent to

$$D_1r_{1i}=d_{2i},\quad i\in\underline{k}. \tag{27}$$

By structural stability, output regulation holds through a nbhd of A_3 in $\mathbf{R}^{n_1n_2}$. This implies, by Lemma 1, that (25), (26), (27) have a solution r_{1i}, $r_{ci}(i\in\underline{k})$ throughout a nbhd of a_{31} in $\mathbf{R}^{n_1}$. Thus for each $\hat{a}_{31}$ in some nbhd of a_{31} there exist $\hat{r}_{1i}\in\mathcal{X}_1$ and $\hat{r}_{ci}\in\mathcal{X}_c\,(i\in\underline{k})$ such that

$$\begin{aligned}(A_1+B_1FC_1-\lambda)\hat{r}_{11}+B_1F_c\hat{r}_{c1}&=\hat{a}_{31}+B_1Fc_{21}\\ B_cC_1\hat{r}_{11}+(A_c-\lambda)\hat{r}_{c1}&=B_cc_{21}\\ B_cC_1\hat{r}_{12}-\hat{r}_{c1}+(A_c-\lambda)\hat{r}_{c2}&=B_cc_{22}\\ &\vdots\\ B_cC_1\hat{r}_{1k}-\hat{r}_{c,k-1}+(A_c-\lambda)\hat{r}_{ck}&=B_cc_{2k}\\ D_1\hat{r}_{1i}&=d_{2i},\quad i\in\underline{k}.\end{aligned}$$

When $\hat{a}_{31}=a_{31}$ denote the corresponding $\hat{r}_{1i}$ and $\hat{r}_{ci}$ by r_{1i}, r_{ci}. Set $\delta a_{31}=\hat{a}_{31}-a_{31}$, $\delta r_{1i}=\hat{r}_{1i}-r_{1i}$, $\delta r_{ci}=\hat{r}_{ci}-r_{ci}$. Then for any δa_{31} in some nbhd of the origin of $\mathbf{R}^{n_1}$

there exist $\delta r_{1i} \in \mathcal{X}_1$ and $\delta r_{ci} \in \mathcal{X}_c$ such that

$$(A_1 + B_1FC_1 - \lambda)\delta r_{11} + B_1F_c\delta r_{c1} = \delta a_{31}$$

$$B_cC_1\delta r_{11} + (A_c - \lambda)\delta r_{c1} = 0$$

$$B_cC_1\delta r_{12} + (A_c - \lambda)\delta r_{c2} = \delta r_{c1}$$

$$\vdots$$

$$B_cC_1\delta r_{1k} + (A_c - \lambda)\delta r_{ck} = \delta r_{c,k-1}$$

$$D_1\delta r_{1i} = 0, \quad i \in \underline{k}.$$

Here a brief computation shows that

$$\delta r_{c1} \in \langle A_c - \lambda | B_cC_1\mathrm{Ker}\,D_1\rangle + \mathrm{Im}(A_c - \lambda)^{k-1}$$

$$= \langle A_c | B_cC_1\mathrm{Ker}\,D_1\rangle + \mathrm{Im}(A_c - \lambda)^{k-1}$$

and hence $(\delta r_{11}, \delta r_{c1}) \in \mathbb{S}$. Thus, since

$$\mathrm{Span}\{\delta a_{31} : \delta a_{31} \text{ in a nbhd of } 0 \in \mathbf{R}^{n_1}\} = \mathcal{X}_1,$$

we find that

$$\mathcal{X}_1 = [A_1 + B_1FC_1 - \lambda, B_1F_c]\mathbb{S}$$

and hence $d(\mathbb{S}) \geq n_1$. ■

4. Necessity of Readability. In this section we prove that readability of z from y is a necessary condition for structural stability. We have first the following partial result.

Proposition 2. *A synthesis is structurally stable relative to A_3 only if (C_1, D_1) is readable.*

Proof. Fix $\lambda \in \sigma(A_2)$ and any prime subspace $\mathcal{X}_{2\lambda}$ of $\mathcal{X}_2$ corresponding to λ. Then $\mathbb{S}$, defined in (22), is contained in

$$\mathrm{Ker}\begin{bmatrix} B_cC_1 & A_c - \lambda \\ D_1 & 0 \end{bmatrix}.$$

Thus Proposition 1 implies

$$d\left(\mathrm{Ker}\begin{bmatrix} B_cC_1 & A_c - \lambda \\ D_1 & 0 \end{bmatrix}\right) \geq n_1$$

and hence

$$d\left(\operatorname{Im}\begin{bmatrix} B_c C_1 & A_c-\lambda \\ D_1 & 0 \end{bmatrix}\right) \leqslant n_c. \tag{28}$$

Now loop stability implies that $A_L-\lambda$ is invertible and thus from (9) the map

$$[B_c C_1, A_c-\lambda] : \mathfrak{X}_1 \oplus \mathfrak{X}_c \to \mathfrak{X}_c \quad \text{is epic.} \tag{29}$$

Then (28) and (29) imply

$$\operatorname{Im}\begin{bmatrix} B_c C_1 & A_c-\lambda \\ D_1 & 0 \end{bmatrix} \approx \operatorname{Im}[B_c C_1, A_c-\lambda]$$

which in turn implies

$$\operatorname{Ker}[B_c C_1, A_c-\lambda] \subset \operatorname{Ker}[D_1, 0]$$

and hence

$$\operatorname{Ker} C_1 \subset \operatorname{Ker} D_1. \quad \blacksquare$$

The main results of this section and the next need the following lemma.

Lemma 2. *Let $A:\mathfrak{X}\to\mathfrak{X}$, $B:\mathfrak{U}\to\mathfrak{X}$ be maps, and $k\geqslant 1$ an integer. Define the subspace $\mathfrak{V}\subset\mathfrak{U}\oplus\mathfrak{X}$ as follows:*

$$\mathfrak{V} = \{(u,x) : Bu + Ax = 0, \quad x \in \langle A|\mathfrak{B}\rangle + \operatorname{Im} A^{k-1}\}.$$

(a). *Let $\bar{A}$ denote the map induced by A in the factor space $\mathfrak{X}/\langle A|\mathfrak{B}\rangle$. Then*

$$d(\mathfrak{V}) \leqslant d(\mathfrak{U}) + d(\operatorname{Ker}\bar{A} \cap \operatorname{Im}\bar{A}^{k-1}). \tag{30}$$

(b). *If $\ell \in \underline{k}$ and*

$$\operatorname{Im} A^{\ell-1} \subset \langle A|\mathfrak{B}\rangle + \operatorname{Im} A^{\ell} \tag{31}$$

then

$$d(\mathfrak{V}) \leqslant d(\mathfrak{U}). \tag{32}$$

Proof.

(a). Suppose B is monic. Then $\mathfrak{U} \approx \mathfrak{B}$ and

$$\mathfrak{V} \approx \tilde{\mathfrak{V}} := A^{-1}\mathfrak{B} \cap (\langle A|\mathfrak{B}\rangle + \operatorname{Im} A^{k-1}). \tag{33}$$

Let $P: \mathcal{X} \to \mathcal{X}/\langle A|\mathcal{B}\rangle$ be the canonical projection. We have

$$d(\mathcal{V}) = d(P\tilde{\mathcal{V}}) + d(\tilde{\mathcal{V}} \cap \operatorname{Ker} P); \tag{34}$$

$$\begin{aligned} P\tilde{\mathcal{V}} &\subset P(A^{-1}\mathcal{B}) \cap P(\langle A|\mathcal{B}\rangle + \operatorname{Im} A^{k-1}) \\ &\subset \operatorname{Ker}\bar{A} \cap \operatorname{Im}\bar{A}^{k-1}; \end{aligned} \tag{35}$$

and

$$\begin{aligned} \tilde{\mathcal{V}} \cap \operatorname{Ker} P &= \tilde{\mathcal{V}} \cap \langle A|\mathcal{B}\rangle \\ &\subset A^{-1}\mathcal{B} \cap \langle A|\mathcal{B}\rangle \\ &\approx \mathcal{B}. \end{aligned} \tag{36}$$

On combining (33)–(36), the inequality (30) follows in this special case. But if B is not monic then $\mathcal{U} \approx \mathcal{B} \oplus \operatorname{Ker} B$, $\mathcal{V} \approx \tilde{\mathcal{V}} \oplus \operatorname{Ker} B$, and the final result follows on addition of $d(\operatorname{Ker} B)$ to both sides of the special inequality just derived.

(b). If

$$\operatorname{Im} A^{\ell-1} \subset \langle A|\mathcal{B}\rangle + \operatorname{Im} A^{\ell}$$

then

$$\operatorname{Im}\bar{A}^{\ell-1} \subset \operatorname{Im}\bar{A}^{\ell}$$

and thus since $\ell \leqslant k$

$$\operatorname{Im}\bar{A}^{k-1} \subset \operatorname{Im}\bar{A}^{k}. \tag{37}$$

If $n := d(\mathcal{X})$ then (37) implies

$$\operatorname{Im}\bar{A}^{k-1} \subset \operatorname{Im}\bar{A}^{n}.$$

So (32) will follow from (30) once we show

$$\operatorname{Ker}\bar{A} \cap \operatorname{Im}\bar{A}^{n} = 0.$$

For a proof by contradiction suppose there exists $\bar{x} \in \operatorname{Ker}\bar{A} \cap \operatorname{Im}\bar{A}^{n}$, $\bar{x} \neq 0$. By the Hamilton-Cayley theorem there exist real scalars a_i such that

$$\bar{A}^{n} = a_0 I + a_1 \bar{A} + \ldots + a_{n-1}\bar{A}^{n-1}.$$

Since $\operatorname{Ker}\bar{A} \neq 0$, $a_0 = 0$. Now there exists $\bar{x}_1$ such that

$$\begin{aligned} \bar{x} &= \bar{A}^{n}\bar{x}_1 \\ &= a_1\bar{A}\bar{x}_1 + \ldots + a_{n-1}\bar{A}^{n-1}\bar{x}_1. \end{aligned} \tag{38}$$

Multiplying (38) by $\bar{A}^{n-1}$, then by $\bar{A}^{n-2}$, etc. yields

$$0=a_1\bar{A}^n\bar{x}_1$$

$$0=a_1\bar{A}^{n-1}\bar{x}_1+a_2\bar{A}^n\bar{x}_1$$

$$\vdots$$

$$0=a_1\bar{A}^2\bar{x}_1+\ldots+a_{n-1}\bar{A}^n\bar{x}_1.$$

Since $\bar{x}\neq 0$ these equations imply in turn $a_1=0$, $a_2=0,\ldots,a_{n-1}=0$. Thus $\bar{A}^n=0$, hence $\bar{x}=0$, a contradiction. ∎

Consider a synthesis which is structurally stable at A_3. By Proposition 2, $\operatorname{Ker} C_1\subset\operatorname{Ker} D_1$; hence, as in (16), $\mathcal{Z}$ can be imbedded in $\mathcal{Y}$. Write

$$\mathcal{Y}=\mathcal{Y}_1\oplus\mathcal{Y}_2 \tag{39}$$

where $\mathcal{Y}_2=\mathcal{Z}$ and $\mathcal{Y}_1$ is an arbitrary complement of $\mathcal{Y}_2$ in $\mathcal{Y}$.

Theorem 1. *(Necessity of readability). A synthesis is structurally stable at $(A_3,B_c|\mathcal{Y}_2)$ only if $([C_1,C_2],[D_1,D_2])$ is readable.*

Proof. Since $\operatorname{Ker} C_1\subset\operatorname{Ker} D_1$ we may write

$$C_1=\begin{bmatrix} E_1\\ D_1\end{bmatrix} \tag{40}$$

for some map $E_1:\mathcal{X}_1\to\mathcal{Y}_1$. Corresponding to the decomposition (39) we may also write

$$C_2=\begin{bmatrix} E_2\\ \tilde{D}_2\end{bmatrix},\quad B_c=[B_{c1},B_{c2}]. \tag{41}$$

We shall have proved

$$\operatorname{Ker}[C_1,C_2]\subset\operatorname{Ker}[D_1,D_2]$$

once we show $\tilde{D}_2=D_2$. Assume, for a proof by contradiction, that

$$\tilde{D}_2\neq D_2. \tag{42}$$

From Lemma 1 structural stability at $(A_3,B_c|\mathcal{Y}_2)$ implies that throughout a nbhd of B_{c2} there exists $R:\mathcal{X}_2\to\mathcal{X}_L$ such that

$$A_LR\dot{-}RA_2=B_L$$

$$D_LR=D_2.$$

From (9) and (10) these equations imply in particular

$$B_cC_1R_1 + A_cR_c - R_cA_2 = B_cC_2 \tag{43a}$$

$$D_1R_1 = D_2. \tag{43b}$$

Now let $\mathcal{X}_{11}$ be an arbitrary complement of $\operatorname{Ker} D_1$ in $\mathcal{X}_1$:

$$\mathcal{X}_1 = \mathcal{X}_{11} \oplus \operatorname{Ker} D_1. \tag{44}$$

Then from (14), D_1 has a right inverse $D_1^\dagger: \mathcal{Z} \to \mathcal{X}_1$ with $\operatorname{Im} D_1^\dagger = \mathcal{X}_{11}$. So $D_1R_1 = D_2$ if and only if $R_1 = \hat{R}_1 + D_1^\dagger D_2$ for some map $\hat{R}_1: \mathcal{X}_2 \to \mathcal{X}_1$ with $\operatorname{Im} \hat{R}_1 \subset \operatorname{Ker} D_1$. Thus throughout a nbhd of B_{c2} there exist $\hat{R}_1: \mathcal{X}_2 \to \mathcal{X}_1$, $R_c: \mathcal{X}_2 \to \mathcal{X}_c$ such that $\operatorname{Im} \hat{R}_1 \subset \operatorname{Ker} D_1$ and, from (43a),

$$B_cC_1\hat{R}_1 + A_cR_c - R_cA_2 = B_c(C_2 - C_1D_1^\dagger D_2). \tag{45}$$

Substitution of (40) and (41) into (45) gives

$$B_{c1}E_1\hat{R}_1 + A_cR_c - R_cA_2 = B_{c1}(E_2 - E_1D_1^\dagger D_2) + B_{c2}(\tilde{D}_2 - D_2). \tag{46}$$

Now throughout a nbhd of B_{c2} (46) has a solution $\hat{R}_1, R_c$ with $\operatorname{Im} \hat{R}_1 \subset \operatorname{Ker} D_1$. Hence, throughout a nbhd of $\delta B_{c2} = 0$, the equation

$$B_{c1}E_1(\delta\hat{R}_1) + A_c(\delta R_c) - (\delta R_c)A_2 = (\delta B_{c2})(\tilde{D}_2 - D_2) \tag{47}$$

has a solution $\delta\hat{R}_1, \delta R_c$ with $\operatorname{Im}(\delta\hat{R}_1) \subset \operatorname{Ker} D_1$.

By assumption (42) there is some $\lambda \in \sigma(A_2)$ and some prime subspace $\mathcal{X}_{2\lambda}$ of $\mathcal{X}_2$ corresponding to λ such that

$$\tilde{D}_2 | \mathcal{X}_{2\lambda} \neq D_2 | \mathcal{X}_{2\lambda}.$$

Fix such λ and let $k = d(\mathcal{X}_{2\lambda})$. Choose a basis in $\mathcal{X}_{2\lambda}$ so that $A_2 | \mathcal{X}_{2\lambda}$ is represented by the matrix $J_k(\lambda)$. In this basis, suppose that the first nonzero column of $(\tilde{D}_2 - D_2) | \mathcal{X}_{2\lambda}$ is the ℓ^{th}, $1 \leqslant \ell \leqslant k$. Restricting equation (47) to $\mathcal{X}_{2\lambda}$, and writing the matrices out explicitly we find that for any $x_c \in \mathcal{X}_c$ there exist $r_{1i} \in \operatorname{Ker} D_1$ and $r_{ci} \in \mathcal{X}_c$ $(i \in \underline{\ell})$ such that

$$\begin{aligned}
B_{c1}E_1r_{11} + (A_c - \lambda)r_{c1} &= 0 \\
B_{c1}E_1r_{12} - r_{c1} + (A_c - \lambda)r_{c2} &= 0 \\
&\vdots \\
B_{c1}E_1r_{1,\ell-1} - r_{c,\ell-2} + (A_c - \lambda)r_{c,\ell-1} &= 0 \\
B_{c1}E_1r_{1\ell} - r_{c,\ell-1} + (A_c - \lambda)r_{c\ell} &= x_c.
\end{aligned} \tag{48}$$

From (48)

$$(A_c-\lambda)^{\ell-1}x_c = B_{c1}E_1r_{11} + (A_c-\lambda)B_{c1}E_1r_{12} + \cdots$$
$$+ (A_c-\lambda)^{\ell-1}B_{c1}E_1r_{1\ell} + (A_c-\lambda)^{\ell}r_{c\ell}.$$

Thus

$$\operatorname{Im}(A_c-\lambda)^{\ell-1} \subset \langle A_c-\lambda | B_{c1}E_1 \operatorname{Ker} D_1\rangle + \operatorname{Im}(A_c-\lambda)^{\ell}$$
$$= \langle A_c | B_{c1}E_1 \operatorname{Ker} D_1\rangle + \operatorname{Im}(A_c-\lambda)^{\ell}. \tag{49}$$

Substitute (40) and (41) into (22) to obtain

$$\mathcal{S} = \big\{(x_1,x_c): B_{c1}E_1x_1 + (A_c-\lambda)x_c = 0, \quad x_1 \in \operatorname{Ker} D_1,$$
$$x_c \in \langle A_c | B_{c1}E_1 \operatorname{Ker} D_1\rangle + \operatorname{Im}(A_c-\lambda)^{k-1}\big\}.$$

Now exploit Lemma 2b by defining

$$A = A_c - \lambda, \qquad B = B_{c1}E_1$$
$$\mathcal{X} = \mathcal{X}_c, \qquad \mathcal{U} = \operatorname{Ker} D_1, \quad \mathcal{V} = \mathcal{S}.$$

Notice that $\langle A|\mathcal{B}\rangle = \langle A-\mu|\mathcal{B}\rangle$ for any $\mu \in \mathbf{C}$. Then (31) follows from (49). Lemma 2b implies

$$d(\mathcal{S}) \leqslant d(\operatorname{Ker} D_1) = n_1 - q$$

which contradicts Proposition 1. Thus (42) is false and $\tilde{D}_2 = D_2$ after all. ∎

5. Necessity of the Internal Model and Feedback. Having proved the necessity of readability (under suitable conditions) we consider in this section only syntheses in which z is readable from y, and hence adopt the representations (17) and (18).

The following characterization of the internal model will be useful. Recall that $k(\lambda)$ is the degree of the factor $s-\lambda$ in the m.p. of A_2.

Lemma 3. *Let $A:\mathcal{X}\to\mathcal{X}$ be a map. Then A incorporates an internal model of A_2 if and only if, for each $\lambda \in \sigma(A_2)$,*

$$d\big[\operatorname{Ker}(A-\lambda) \cap \operatorname{Im}(A-\lambda)^{k(\lambda)-1}\big] \geqslant q.$$

Proof. Fix $\lambda \in \sigma(A_2)$ and consider a Jordan decomposition of $\mathcal{X}$ relative to A. Let $\mu \in \sigma(A)$ and let $\mathcal{X}_\mu$ be a prime subspace corresponding to μ in this decomposition. Finally, define $A_\mu := A|\mathcal{X}_\mu$ and $\ell := d(\mathcal{X}_\mu)$. Then there exists a cyclic generator $x \in \mathcal{X}_\mu$ such that

$$\mathcal{X}_\mu = \operatorname{Span}\left\{x, (A_\mu - \mu)x, \ldots, (A_\mu - \mu)^{\ell-1}x\right\}$$

$$\operatorname{Im}(A_\mu - \mu) = \operatorname{Span}\left\{(A_\mu - \mu)x, \ldots, (A_\mu - \mu)^{\ell-1}x\right\}$$

$$\operatorname{Ker}(A_\mu - \mu) = \operatorname{Span}\left\{(A_\mu - \mu)^{\ell-1}x\right\}.$$

Hence if i is an integer then

$$d\left[\operatorname{Ker}(A_\mu - \mu) \cap \operatorname{Im}(A_\mu - \mu)^{i-1}\right] = \begin{cases} 1 & \text{if } i \in \underline{\ell} \\ 0 & \text{if } i > \ell. \end{cases}$$

From this it is seen that

$$d\left[\operatorname{Ker}(A - \lambda) \cap \operatorname{Im}(A - \lambda)^{k(\lambda)-1}\right]$$

equals the number of prime subspaces $\mathcal{X}_\lambda \subset \mathcal{X}$ corresponding to λ such that $d(\mathcal{X}_\lambda) \geq k(\lambda)$, and hence equals the number of i.f. of A divisible by $(s-\lambda)^{k(\lambda)}$. ■

For the main result of this section we shall assume that

$$\operatorname{Im} B_{cw} \subset \langle A_c | B_{cw} E_1 \operatorname{Ker} D_1 \rangle \tag{50}$$

or equivalently

$$\langle A_c | \operatorname{Im} B_{cw} \rangle = \langle A_c | B_{cw} E_1 \operatorname{Ker} D_1 \rangle.$$

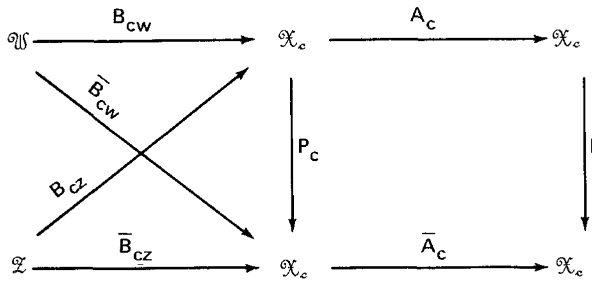
A systemic interpretation of (50) is the following: the information carried by $w(\cdot)$ which is processed by the compensator pertains only to the plant and is not derivable from $z(\cdot)$. It will be shown in Proposition 3 below that (50) is a necessary condition for structural stability.

Theorem 2. (*Necessity of the internal model and feedback*). *Consider a synthesis in which (50) holds. This synthesis is structurally stable at A_3 only if the compensator incorporates an internal model of A_2 which is controllable by z and observable by u.*

Define

$$\mathcal{R}_c = \langle A_c | B_{cw} E_1 \operatorname{Ker} D_1 \rangle. \tag{51}$$

Let $P_c : \mathcal{X}_c \to \overline{\mathcal{X}}_c := \mathcal{X}_c / \mathcal{R}_c$ be the canonical projection and define $\overline{A}_c$, $\overline{B}_{cw}$, $\overline{B}_{cz}$ by the commutative diagram below.



Proof of Theorem 2. We shall show that $\bar{A}_c$ incorporates an internal model of A_2; further

$$\bar{\mathfrak{X}}_c = \operatorname{Im}(\bar{A}_c - \lambda) + \operatorname{Im} \bar{B}_{cz}, \quad \lambda \in \sigma(A_2)$$

and

$$\operatorname{Ker} F_c \cap \operatorname{Ker}(A_c - \lambda) = 0, \quad \lambda \in \sigma(A_2).$$

In view of (50) and (51) the proof will then be complete.

Fix $\lambda \in \sigma(A_2)$ and define $\mathbb{S}$ by (22) taking $k = k(\lambda)$. From (17) and (18)

$$\mathbb{S} = \{(x_1, x_c) : B_{cw} E_1 x_1 + (A_c - \lambda) x_c = 0, \quad x_1 \in \operatorname{Ker} D_1,$$

$$x_c \in \langle A_c | B_{cw} E_1 \operatorname{Ker} D_1 \rangle + \operatorname{Im}(A_c - \lambda)^{k-1}\}.$$

Now appealing to Lemma 2a with

$$A = A_c - \lambda, \quad B = B_{cw} E_1$$

$$\mathfrak{X} = \mathfrak{X}_c, \quad \mathfrak{U} = \operatorname{Ker} D_1, \quad \mathcal{V} = \mathbb{S}$$

we find

$$d(\mathbb{S}) \leqslant d(\operatorname{Ker} D_1) + d\left[\operatorname{Ker}(\bar{A}_c - \lambda) \cap \operatorname{Im}(\bar{A}_c - \lambda)^{k-1}\right]. \tag{52}$$

Since from Proposition 1 $d(\mathbb{S}) \geqslant n_1$ and since $d(\operatorname{Ker} D_1) = n_1 - q$, (52) implies

$$d\left[\operatorname{Ker}(\bar{A}_c - \lambda) \cap \operatorname{Im}(\bar{A}_c - \lambda)^{k-1}\right] \geqslant q.$$

Thus from Lemma 3 $\bar{A}_c$ incorporates an internal model of A_2.

Loop stability implies that $A_L - \lambda$ is invertible. Thus from (9)

$$\operatorname{Ker}(B_1 F_c) \cap \operatorname{Ker}(A_c - \lambda) = 0 \tag{53}$$

and

$$\mathfrak{X}_c = \operatorname{Im}(A_c - \lambda) + \operatorname{Im}(B_c C_1). \tag{54}$$

From (53)

$$\operatorname{Ker} F_c \cap \operatorname{Ker}(A_c - \lambda) = 0$$

and hence the internal model is observable by u. From (54)

$$\mathcal{X}_c = \operatorname{Im}(A_c - \lambda) + \operatorname{Im} B_c$$

which implies, from (18),

$$\mathcal{X}_c = \operatorname{Im}(A_c - \lambda) + \operatorname{Im} B_{cw} + \operatorname{Im} B_{cz}. \tag{55}$$

Now from (50) $\bar{B}_{cw} = 0$; so when P_c is applied to both sides of (55) there results

$$\bar{\mathcal{X}}_c = \operatorname{Im}(\bar{A}_c - \lambda) + \operatorname{Im} \bar{B}_{cz};$$

that is, the internal model is controllable by z. ∎

The purpose of the internal model can be understood in the frequency domain by reference to Fig. 2. Let $G_L(s)$ denote the transfer matrix from the x_2 terminal to the z terminal:

$$G_L(s) = D_L(s - A_L)^{-1} B_L + D_2.$$

The internal model supplies transmission zeros of $G_L(s)$ to cancel the poles of $(s - A_2)^{-1}$. Loop transmission zeros and their relation to the internal model are treated in detail in a subsequent article [5].

Let κ be the degree of the m.p. of A_2. If $\bar{A}_c$ incorporates an internal model of A_2 its order is at least $q\kappa$; that is $d(\bar{\mathcal{X}}_c) \geqslant q\kappa$. In [1, Chapter 8] the problem was solved of constructing a structurally stable synthesis; it was shown that such a synthesis exists with map $\bar{A}_c$ incorporating an internal model of order precisely $q\kappa$. We conclude that $q\kappa$ is the minimal order of internal model necessary for structural stability of the type considered here. If a weaker version of structural stability is required, for example if some of the elements of A_3 are fixed, then usually a complete internal model of the kind described here is not necessary.

We remark that the choice of data point A_3 in Theorem 2 is not crucial, but seems to allow a proof which is simpler than one based on alternative data points.

Assumption (50) is equivalent to the assumption that $\bar{B}_{cw} = 0$. This in turn is equivalent to the assumption that, in Fig. 4, there is no direct signal flow from w to v. That (50) is a necessary requirement for structural stability is established by the following.

Proposition 3. *There is no synthesis in which (50) fails and which is structurally stable at* $(A_3, \bar{B}_{cw})$.

Proof. For a proof by contradiction assume there is such a synthesis. Then from Lemma 1, throughout a nbhd of $\bar{B}_{cw}$ there exists $R: \mathcal{X}_2 \to \mathcal{X}_L$ such that

$$A_L R - R A_2 = B_L$$

$$D_L R = D_2.$$

This implies, from (9) and (10), that throughout a nbhd of $\bar{B}_{cw}$ there exist $R_1: \mathcal{X}_2 \to \mathcal{X}_1$ and $R_c: \mathcal{X}_2 \to \mathcal{X}_c$ such that

$$B_c C_1 R_1 + A_c R_c - R_c A_2 = B_c C_2$$

$$D_1 R_1 = D_2,$$

or, from (17) and (18),

$$B_{cw} E_1 R_1 + A_c R_c - R_c A_2 = B_c E_2 \tag{56a}$$

$$D_1 R_1 = D_2. \tag{56b}$$

As in (44) let $\mathcal{X}_{11}$ be an arbitrary complement of $\operatorname{Ker} D_1$ in $\mathcal{X}_1$ and let $D_1^\dagger: \mathfrak{Z} \to \mathcal{X}_1$ be a right inverse of D_1 with $\operatorname{Im} D_1^\dagger = \mathcal{X}_{11}$. Then (56b) is equivalent to $R_1 = \hat{R}_1 + D_1^\dagger D_2$ for some map $\hat{R}_1: \mathcal{X}_2 \to \mathcal{X}_1$ with $\operatorname{Im} \hat{R}_1 \subset \operatorname{Ker} \mathrm{D}_1$, so (56a) becomes

$$B_{cw} E_1 \hat{R}_1 + A_c R_c - R_c A_2 = B_{cw}(E_2 - E_1 D_1^\dagger D_2). \tag{57}$$

Apply P_c to (57) to give

$$\bar{A}_c \bar{R}_c - \bar{R}_c A_2 = \bar{B}_{cw}(E_2 - E_1 D_1^\dagger D_2). \tag{58}$$

Here $\bar{R}_c := P_c R_c$ and we have used the fact that $P_c B_{cw} E_1 \operatorname{Ker} D_1 = 0$. Now (58) has a solution $\bar{R}_c: \mathcal{X}_2 \to \bar{\mathcal{X}}_c$ throughout a nbhd of $\bar{B}_{cw}$.

Since by (13) $[C_1, C_2]$ is epic, for each $w \in \mathcal{W}$ there exist $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ such that

$$w = E_1 x_1 + E_2 x_2$$

$$0 = D_1 x_1 + D_2 x_2;$$

equivalently, for every $w \in \mathcal{W}$ there exist $\hat{x}_1 \in \operatorname{Ker} D_1$ and $x_2 \in \mathcal{X}_2$ such that

$$w = E_1 \hat{x}_1 + (E_2 - E_1 D_1^\dagger D_2) x_2.$$

Thus

$$\mathcal{W} = E_1 \operatorname{Ker} D_1 + \operatorname{Im}(E_2 - E_1 D_1^\dagger D_2)$$

and hence

$$\operatorname{Im} \bar{B}_{cw} = \bar{B}_{cw} \operatorname{Im}(E_2 - E_1 D_1^\dagger D_2). \tag{59}$$

Since (50) is assumed to fail $\bar{B}_{cw} \neq 0$, and so (59) implies that

$$E_2 - E_1 D_1^{\dagger} D_2 \neq 0.$$

Hence there exist $\lambda \in \sigma(A_2)$ and a prime subspace $\mathcal{X}_{2\lambda}$ of $\mathcal{X}_2$ corresponding to λ such that $(E_2 - E_1 D_1^{\dagger} D_2) | \mathcal{X}_{2\lambda} \neq 0$. Fix a basis for $\mathcal{X}_{2\lambda}$ so that A_2 is represented by $J_k(\lambda)$ where $k := d(\mathcal{X}_{2\lambda})$. Suppose the first nonzero column of $(E_2 - E_1 D_1^{\dagger} D_2) | \mathcal{X}_{2\lambda}$ is the ℓ^{th}, $\ell \in \underline{k}$. Then, restricting (58) to $\mathcal{X}_{2\lambda}$ and writing the matrices out explicitly we find that for any $\bar{x}_c \in \bar{\mathcal{X}}_c$ there exist $\bar{r}_{ci} \in \bar{\mathcal{X}}_c$ $(i \in \underline{\ell})$ such that

$$
\begin{aligned}
(\bar{A}_c - \lambda)\bar{r}_{c1} &= 0 \\
(\bar{A}_c - \lambda)\bar{r}_{c2} &= \bar{r}_{c1} \\
&\vdots \\
(\bar{A}_c - \lambda)\bar{r}_{c,\ell-1} &= \bar{r}_{c,\ell-2} \\
(\bar{A}_c - \lambda)\bar{r}_{c\ell} &= \bar{r}_{c,\ell-1} + \bar{x}_c.
\end{aligned}
\tag{60}
$$

Now (60) implies that for any $\bar{x}_c$ there exists $\bar{r}_{cl}$ such that

$$(\bar{A}_c - \lambda)^{\ell} \bar{r}_{c\ell} = (\bar{A}_c - \lambda)^{\ell - 1} \bar{x}_c.$$

Thus

$$\operatorname{Im}(\bar{A}_c - \lambda)^{\ell - 1} \subset \operatorname{Im}(\bar{A}_c - \lambda)^{\ell}. \tag{61}$$

Next apply Lemma 2b with

$$A = \bar{A}_c - \lambda, \quad B = 0, \quad k = k(\lambda),$$

$$\mathcal{X} = \bar{\mathcal{X}}_c, \quad \mathcal{U} = 0.$$

Then (31) follows from (61) and so (32) implies

$$\operatorname{Ker}(\bar{A}_c - \lambda) \cap \operatorname{Im}(\bar{A}_c - \lambda)^{k(\lambda) - 1} = 0.$$

By Lemma 3 this implies that $\bar{A}_c$ does *not* incorporate an internal model of A_2; but this contradicts Theorem 2. ■

To further explain the significance of condition (50) we shall give an intuitive argument to indicate why Proposition 3 ought to be true. Assume for simplicity that $E_1 = 0$ and $F = 0$. Then Fig. 3 reduces to Fig. 5.

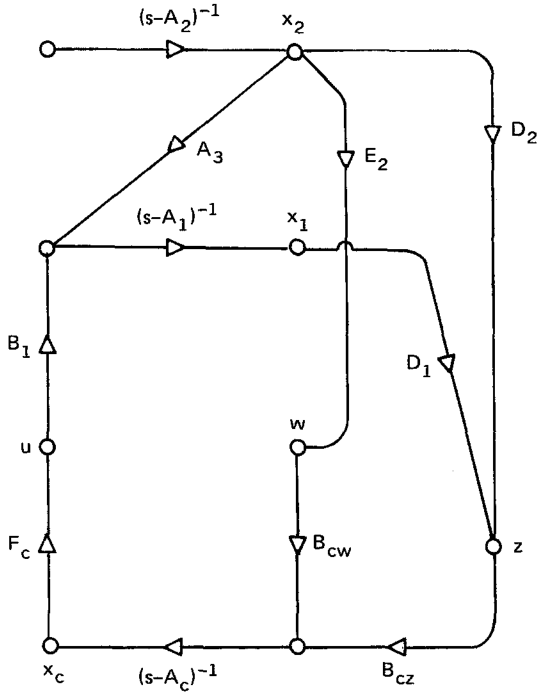


Figure 5. Figure 3 with $E_1=0$, $F=0$.

Suppose the synthesis is structurally stable at A_3 so that, by Theorem 2, the compensator incorporates an internal model of A_2. Suppose also that (50) fails, namely $B_{cw}\neq 0$. The compensator can be thought of as acting in the following way. On the basis of information about z, the internal model injects into the loop signals which (asymptotically) counterbalance the disturbances entering via A_3 and D_2. The internal model is activated by information about z via B_{cz}. Since $E_1=0$ and $[C_1, C_2]$ is epic, E_2 is epic. Thus for some initial condition $x_2(0)$ the signal $w(t)=E_2x_2(t)$ will not converge to 0 as $t\to\infty$. Hence, possibly after a small perturbation of B_{cw}, the internal model processes, not the 'accurate' information $B_{cz}z(\cdot)$, but instead a 'noisy' signal $B_{cz}z(\cdot)+B_{cw}w(\cdot)$. Thus in the structure of Fig. 5 any useful rôle played by the signal path $x_2\to w\to x_c$ is (generally) vitiated by a small perturbation in B_{cw}.

6. Conclusion. Synthesis procedures for achieving structurally stable controllers in the linear multivariable setting have been given by Davison [3], Pearson et al. [6] and Wonham [1]. The present paper has considered the converse problem: What controller properties are necessary for structural stability? Our major conclusion can be summarized as *The Internal Model Principle*: *A regulator synthesis is structurally stable only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process.*

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